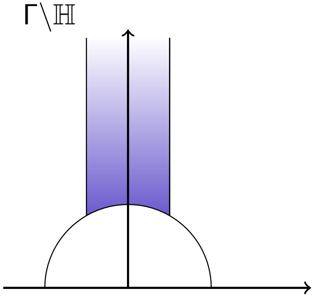
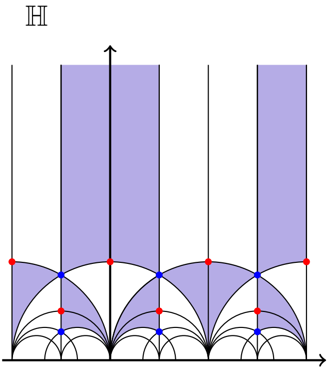
Some applications of Modular Forms

- $\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, the Upper Half Plane
- $\mathrm{SL}(2,\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_{2\times 2}(\mathbb{Z}) \mid ad - bc = 1 \right\}$
- $\Gamma = \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm \mathrm{id}\}$, The Modular group.
- Γ acts on $\mathbb{H}$ by linear fractional transformations;

$$\mathbb{H} \ni \tau \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d} \in \mathbb{H} \quad \left(\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma\right)$$

The fundamental domain $\Gamma\backslash\mathbb{H}$ and the tiling of $\mathbb{H}$





Quadratic forms

- For $\mathbb{Z} \ni D \equiv 0, 1 \pmod 4$, let $\mathcal{Q}_D$ be the set of integral binary quadratic forms of discriminant D that are positive definite if $D < 0$.

$$\mathcal{Q}_D := \{Q = [A, B, C] \mid A, B, C \in \mathbb{Z}, B^2 - 4AC = D\}$$

  where $[A, B, C] := Q(x, y) = Ax^2 + Bxy + Cy^2$

- $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ acts also on $\mathcal{Q}_D$.
- For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $Q \in \mathcal{Q}_D$

$$(\gamma Q)(x, y) := Q(dx - by, -cx + ay)$$

- This action is compatible with the action of Γ on the roots τ of $Q(\tau, 1) = 0$ by linear fractional transformation.

Quadratic forms

- The set of classes $\Gamma\backslash\mathcal{Q}_D$ is finite.
- $|\Gamma\backslash\mathcal{Q}_D| := h(D)$ is called the class number.
- The classes of primitive forms $((A, B, C) = 1)$ form an abelian group, called the class group
- $D = -191 \;\; h(D) = 13$
  $\Gamma\backslash\mathcal{Q}_D = \{[1, 1, 48], [2, 1, 242], [2, -1, 24], [3, 1, 16], [3, -1, 16], [4, 1, 12], [4, -1, 12], [6, 1, 8], [6, -1, 8], [5, 3, 10], [5, -3, 10], [6, 5, 9], [6, -5, 9]\}$
- $D = 28, \;\; h(D) = 2$
  $\Gamma\backslash\mathcal{Q}_D = \{[1, 4, -3], [-1, 4, 3]\}$

Quadratic forms

- For $Q \in \mathcal{Q}_D$, the isotropy group $\Gamma_Q = \{\gamma \in \Gamma; \gamma Q = Q\}$ consists of all transformations

$$\gamma = \pm \begin{pmatrix} \frac{t+Bu}{2} & Cu \\ -Au & \frac{t-Bu}{2} \end{pmatrix}$$

  where (t, u) are positive integral solutions of the Pell equation $t^2 - Du^2 = 4$.
- If $D < 0$, then $\Gamma_Q = \{\mathrm{id}\}$ unless $D = -3, -4$, in which case it has order 3 or 2, respectively.
- If $D > 0$, then $\Gamma_Q = < g_Q >$ is infinite cyclic with generator $\gamma = g_Q$ coming from minimal $t, u > 0$.

$D < 0$

- For a given quadratic form $Q = [A, B, C]$, we associate a complex number $\tau_Q = \frac{-B+\sqrt{D}}{2A} \in \mathbb{H}$.
- Given two equivalent quadratic forms,
  $[A, B, C] = Q \sim Q' = [a, b, c] \in \mathcal{Q}_D$,

  we can get two equivalent complex numbers,
  $\tau_Q = \frac{-B+\sqrt{D}}{2A}$ and $\tau_{Q'} = \frac{-b+\sqrt{D}}{2a} \in \mathbb{H}$.

$D < 0$

- For a given quadratic form $Q = [A, B, C]$, we associate a complex number $\tau_Q = \frac{-B+\sqrt{D}}{2A} \in \mathbb{H}$.
- Given two equivalent quadratic forms,
  $[A, B, C] = Q \sim Q' = [a, b, c] \in \mathcal{Q}_D$,

  we can get two equivalent complex numbers,
  $\tau_Q = \frac{-B+\sqrt{D}}{2A}$ and $\tau_{Q'} = \frac{-b+\sqrt{D}}{2a} \in \mathbb{H}$.

  $\tau_Q \sim \tau_{Q'}$ in the sense that $\tau_Q = \gamma\tau_{Q'}$ for some $\gamma \in \Gamma$.

$D < 0$

- For a given quadratic form $Q = [A, B, C]$, we associate a complex number $\tau_Q = \frac{-B+\sqrt{D}}{2A} \in \mathbb{H}$.
- Given two equivalent quadratic forms,
  $[A, B, C] = Q \sim Q' = [a, b, c] \in \mathcal{Q}_D$,

  we can get two equivalent complex numbers,
  $\tau_Q = \frac{-B+\sqrt{D}}{2A}$ and $\tau_{Q'} = \frac{-b+\sqrt{D}}{2a} \in \mathbb{H}$.

  $\tau_Q \sim \tau_{Q'}$ in the sense that $\tau_Q = \gamma\tau_{Q'}$ for some $\gamma \in \Gamma$.
- This way each $[Q] \in \Gamma\backslash\mathcal{Q}_D$ corresponds to a CM point $\tau_Q \in \Gamma\backslash\mathbb{H}$.

$D < 0$

- For a given quadratic form $Q = [A, B, C]$, we associate a complex number $\tau_Q = \frac{-B+\sqrt{D}}{2A} \in \mathbb{H}$.
- Given two equivalent quadratic forms,
  $[A, B, C] = Q \sim Q' = [a, b, c] \in \mathcal{Q}_D$,

  we can get two equivalent complex numbers,
  $\tau_Q = \frac{-B+\sqrt{D}}{2A}$ and $\tau_{Q'} = \frac{-b+\sqrt{D}}{2a} \in \mathbb{H}$.

  $\tau_Q \sim \tau_{Q'}$ in the sense that $\tau_Q = \gamma\tau_{Q'}$ for some $\gamma \in \Gamma$.
- This way each $[Q] \in \Gamma\backslash\mathcal{Q}_D$ corresponds to a CM point $\tau_Q \in \Gamma\backslash\mathbb{H}$.
- $D = -4,\ h(D) = 1,\ \Gamma\backslash\mathcal{Q}_D = \{Q = [1, 0, 1]\},\ \tau_Q = \sqrt{-1}$.

What can we say about h_D?

How many $\tau_Q \in \Gamma\backslash\mathbb{H}$ *are there?*

What can we say about h_D?

How many $\tau_Q \in \Gamma\backslash\mathbb{H}$ are there?

- Gauss' conjecture: $h(D) \to \infty$ as $D \to -\infty$

What can we say about h_D?

How many $\tau_Q \in \Gamma\backslash\mathbb{H}$ *are there?*

- Gauss' conjecture: $h(D) \to \infty$ as $D \to -\infty$
  - Hecke (1918): If Generalized Riemann Hypothesis is true then $h(D) \to \infty$ as $D \to -\infty$

What can we say about h_D?

How many $\tau_Q \in \Gamma\backslash\mathbb{H}$ are there?

- Gauss' conjecture: $h(D) \to \infty$ as $D \to -\infty$
  - Hecke (1918): If Generalized Riemann Hypothesis is true then $h(D) \to \infty$ as $D \to -\infty$
  - Deuring (1933): If Riemann Hypothesis is false then $h(D) \geq 2$ for all $-D$ sufficiently large

What can we say about h_D?

How many $\tau_Q \in \Gamma\backslash\mathbb{H}$ are there?

- Gauss' conjecture: $h(D) \to \infty$ as $D \to -\infty$
  - Hecke (1918): If Generalized Riemann Hypothesis is true then $h(D) \to \infty$ as $D \to -\infty$
  - Deuring (1933): If Riemann Hypothesis is false then $h(D) \geq 2$ for all $-D$ sufficiently large
  - Mordell, Heilbronn(1934): If GRH is false then $h(D) \to \infty$ as $D \to -\infty$
- Theorem: $h(D) \to \infty$ as $D \to -\infty$

What can we say about h_D?

How many $\tau_Q \in \Gamma\backslash\mathbb{H}$ are there?

- Gauss' conjecture: $h(D) \to \infty$ as $D \to -\infty$
  - Hecke (1918): If Generalized Riemann Hypothesis is true then $h(D) \to \infty$ as $D \to -\infty$
  - Deuring (1933): If Riemann Hypothesis is false then $h(D) \geq 2$ for all $-D$ sufficiently large
  - Mordell, Heilbronn(1934): If GRH is false then $h(D) \to \infty$ as $D \to -\infty$
- Theorem: $h(D) \to \infty$ as $D \to -\infty$

- **Theorem:** *(Siegel) For every* $\epsilon > 0$*, there exists a constant* c *which cannot be effectively computed such that* $h(D) > c|D|^{1/2-\epsilon}$

- Theorem: *(Siegel) For every* $\epsilon > 0$, *there exists a constant* c *which cannot be effectively computed such that* $h(D) > c|D|^{1/2-\epsilon}$
- *Gauss class number one problem for* $D < 0$:

- Theorem: *(Siegel) For every* $\epsilon > 0$*, there exists a constant c which cannot be effectively computed such that* $h(D) > c|D|^{1/2-\epsilon}$
- *Gauss class number one problem for* $D < 0$:
  - *Theorem* (Heilbronn-Linfoot(1934)) *There are at most ten negative fundamental discriminants* $D < 0$ *for which* $h(D) = 1$; $D = -3, -4, -7, -8, -11, -19, -43, -67, -163, ?$

- Theorem: *(Siegel) For every $\epsilon > 0$, there exists a constant c which cannot be effectively computed such that* $h(D) > c|D|^{1/2-\epsilon}$
- *Gauss class number one problem for $D < 0$:*
  - *Theorem* (Heilbronn-Linfoot(1934)) *There are at most ten negative fundamental discriminants $D < 0$ for which $h(D) = 1$; $D = -3, -4, -7, -8, -11, -19, -43, -67, -163, ?$*
  - *Theorem* (Baker(1966),Stark(1967), Heegner (1952))

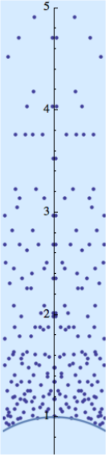
$$h(D) = 1 \iff D = -3, -4, -7, -8, -11, -19, -43, -67, -163.$$

- Theorem: *(Siegel) For every $\epsilon > 0$, there exists a constant c which cannot be effectively computed such that $h(D) > c|D|^{1/2-\epsilon}$*
- *Gauss class number one problem for $D < 0$:*
  - *Theorem (Heilbronn-Linfoot(1934)) There are at most ten negative fundamental discriminants $D < 0$ for which $h(D) = 1$; $D = -3, -4, -7, -8, -11, -19, -43, -67, -163, ?$*
  - *Theorem (Baker(1966),Stark(1967), Heegner (1952))*

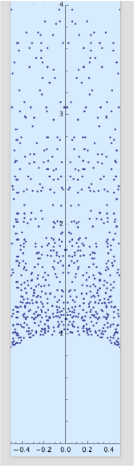
$$h(D) = 1 \iff D = -3, -4, -7, -8, -11, -19, -43, -67, -163.$$

- *Theorem (Goldfeld(1976), Gross-Zagier (1983)) For every $\epsilon > 0$, there exists an effectively computable constant c such that $h(D) > c(\log|D|)^{1-\epsilon}$.*

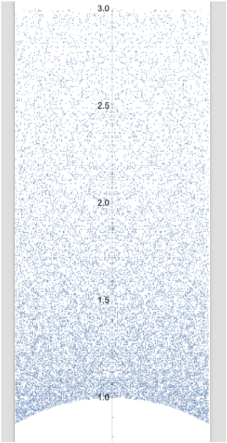
$D = -51071$ $h = 273.$



$D = -1299743 \qquad h = 945.$



$D = -573259391$ $h = 34125.$



Theorem (W.Duke)

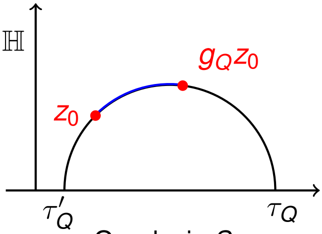
For $D < 0$ be a fundamental discriminant, let $\Lambda_D = \Gamma \backslash \mathcal{Q}_D$. Then the set Λ_D becomes uniformly distributed in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ as $D \to \infty$. That is, given $\phi \in C_c^\infty(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ we have

$$\frac{1}{h(D)} \sum_{\tau_Q \in \Lambda_D} \phi(\tau_Q) \to \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \phi(\tau) d\mu(\tau)$$

where $d\mu(\tau)$ is the $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$

What if $D > 0$?

The root τ_Q of $Q(\tau, 1) = 0$ for $Q \in \mathcal{Q}_D$ is a real quadratic irrationality.
For $D > 0$, non-square, to each $Q = [A, B, C] \in \mathcal{Q}_D$, we can associate a geodesic $S_Q = A|\tau|^2 + B\,\mathrm{Re}(\tau) + C \in \mathbb{H}$ and a closed geodesic $C_Q \in \Gamma\backslash\mathbb{H}$.



Geodesic S_Q

For $D > 0$, let t, u be the smallest positive integers for which $t^2 - Du^2 = 4$ and $\varepsilon_D = \frac{1}{2}(t + u\sqrt{D})$.

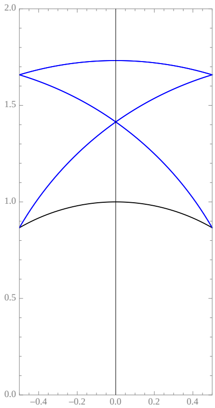
For any $Q \in \mathcal{Q}_D$ we have

$$\int_{C_Q} 1 ds(\tau) = \int_{C_Q} 1 \frac{d\tau}{Q(\tau, 1)} = \log \varepsilon_D$$

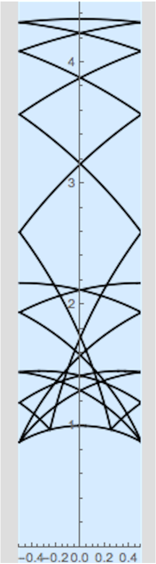
where $ds(\tau)$ is the hyperbolic arc length.

$$\sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \int_{C_Q} 1 ds(\tau) = h_D \log \varepsilon_D.$$

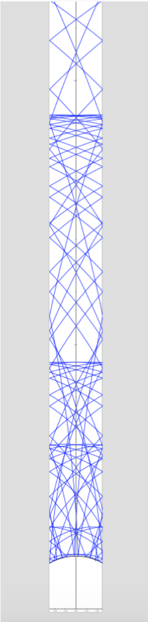
$D = 12$



$D = 28$ $h = 2$



$D = 4x787 \qquad h = 1$



What can we say about $h(D)$, the number of geodesics?

Class number one problem for real quadratic fields:

$$h(D) = 1 \text{ for infinitely many } D > 0.$$

What can we say about $h(D)$, the number of geodesics?

Class number one problem for real quadratic fields:

$$h(D) = 1 \text{ for infinitely many } D > 0.$$

This is completely open.

What can we say about the total length of the geodesics?

What can we say about the total length of the geodesics? $h_D \log \varepsilon_D$?

Theorem (Siegel)

For every $\epsilon > 0$, there exists a constant c which cannot be effectively computed such that $h(D) \log \varepsilon_D > cD^{1/2-\epsilon}$.

Theorem (Duke)

For $D > 0$ be a fundamental discriminant, let $\Lambda_D = \Gamma\backslash\mathcal{Q}_D$ be the set of closed geodesics. Then the set Λ_D becomes uniformly distributed in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ as $D \to \infty$.
That is, given $\phi \in C_c^\infty(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ we have

$$\frac{1}{h(D)\log\varepsilon_D}\sum_{C_Q\in\Lambda_D}\int_{C_Q}\phi(\tau)ds(\tau) \to \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}\phi(\tau)d\mu(\tau)$$

Linnik, Einsiedler, Lindenstrauss, Michel, Venkatesh...

How does W. Duke prove his theorems?

He proved that " Weyl sums" are small.

If $u : \mathbb{H} \longrightarrow \mathbb{C}$ is a Maass cusp form

$$\frac{1}{h(D)} \sum_{\tau_Q \in \Lambda_D} \frac{1}{w_Q} u(\tau_Q) \to 0, \;\; 0 > D \to -\infty$$

$$\frac{1}{h(D) \log \varepsilon_D} \sum_{C_Q \in \Lambda_D} \int_{C_Q} u(\tau) ds(\tau) \to 0, \;\; 0 < D \to \infty$$

H. Weyl's theorem on uniform distribution in $\mathbb{R}/\mathbb{Z}$.

We say a sequence of points $\{x_n\}_{n=0}^{\infty}$ is **uniformly distributed** in $[0, 1]$ if for all $0 \le a \le b \le 1$,

$$\lim_{N\to\infty} \frac{\#\{n \le N \mid a \le x_n \le b\}}{N} = b - a.$$

H. Weyl's theorem on uniform distribution in $\mathbb{R}/\mathbb{Z}$.

We say a sequence of points $\{x_n\}_{n=0}^{\infty}$ is **uniformly distributed** in $[0,1]$ if for all $0 \leq a \leq b \leq 1$,

$$\lim_{N\to\infty} \frac{\#\{n \leq N \mid a \leq x_n \leq b\}}{N} = b - a.$$

Note this says that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx$$

holds with $f = \chi_{[a,b]}$, the characteristic function of $[a,b]$.

Theorem (H.Weyl)

The sequence $\{x_n\}$ *is uniformly distributed in* $\mathbb{R}/\mathbb{Z}$ *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(mx_n) = \int_0^1 e(mx)dx = 0, \ \ \forall m \in \mathbb{Z}\backslash\{0\}$$

To prove

$$\frac{1}{h(D)} \sum_{\tau_Q \in \Lambda_D} \frac{1}{w_Q} u(\tau_Q) \to 0, \;\; 0 > D \to -\infty$$

one needs to give a good bound for $\mathrm{Tr}_D(u) := \sum_{\tau_Q \in \Lambda_D} \frac{1}{w_Q} u(\tau_Q)$ which beats the Siegel bound for $h(D)$.

$$h(D) > c|D|^{1/2-\epsilon}, \;\; \text{if} \;\; D < 0$$

How can one prove a good bound for the Weyl sum $\mathrm{Tr}_D(u)$?

1. Relate $\mathrm{Tr}_D(u)$ to the D-th Fourier coefficients of a half integral weight Maass form $\tilde{u}$ using results of Maass, Katok-Sarnak.

2. Use deep theorems of H. Iwaniec (extended by W. Duke) which give good bounds for the Fourier coefficients of half-integral weight forms.

What can we say about the values of modular functions at CM points ?

Let

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

be the modular invariant for $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$.

What do we know about the individual values of j at CM points, the **Singular moduli**

$$j(\tau_Q) = ?$$

or their sum

$$\sum_{\tau_Q \in \Gamma \backslash \mathcal{Q}_D} j(\tau_Q) = ?$$

Values of modular functions at complex quadratic irrationalities

Theory of complex multiplication says

Values of modular functions at complex quadratic irrationalities

Theory of complex multiplication says

- The singular modulus $j(\tau_Q)$ depends only on the Γ-equivalence class of Q

Values of modular functions at complex quadratic irrationalities

Theory of complex multiplication says

- The singular modulus $j(\tau_Q)$ depends only on the Γ-equivalence class of Q
- Each of the $h(D) = |\Gamma \backslash \mathcal{Q}_D|$ values $j(\tau_Q)$ is an algebraic integer of exact degree $h(D)$

Values of modular functions at complex quadratic irrationalities

Theory of complex multiplication says

- The singular modulus $j(\tau_Q)$ depends only on the Γ-equivalence class of Q
- Each of the $h(D) = |\Gamma \backslash \mathcal{Q}_D|$ values $j(\tau_Q)$ is an algebraic integer of exact degree $h(D)$
- They form a full set of Galois conjugates so that the sum of these values is the algebraic trace.

- For $D < 0$, we define the (modified) trace of singular moduli as

$$\mathrm{Tr}_D(j_1) := \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \frac{1}{|\Gamma_Q|} j_1(\tau_Q)$$

where $j_1(\tau) = j(\tau) - 744$

- For $D < 0$, we define the (modified) trace of singular moduli as

$$\mathrm{Tr}_D(j_1) := \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \frac{1}{|\Gamma_Q|} j_1(\tau_Q)$$

where $j_1(\tau) = j(\tau) - 744$

- In general for $f \in \mathcal{M}_0^!(\Gamma)$, define

$$\mathrm{Tr}_D(f) := \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \frac{1}{|\Gamma_Q|} f(\tau_Q)$$

- For $D < 0$, we define the (modified) trace of singular moduli as

$$\mathrm{Tr}_D(j_1) := \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \frac{1}{|\Gamma_Q|} j_1(\tau_Q)$$

where $j_1(\tau) = j(\tau) - 744$

- In general for $f \in \mathcal{M}_0^!(\Gamma)$, define

$$\mathrm{Tr}_D(f) := \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \frac{1}{|\Gamma_Q|} f(\tau_Q)$$

$$\mathrm{Tr}_D(1) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \frac{1}{|\Gamma_Q|} = H(|D|),$$ *Hurwitz class number*

$$H(0) := \frac{-1}{12}$$

Traces of modular functions

m	$\mathrm{Tr}_{-3}(j_m)$	$\mathrm{Tr}_{-4}(j_m)$	$\mathrm{Tr}_{-7}(j_m)$
0	1/3	1/2	1
1	-248	492	-4119
2	53256	287244	16572393
3	-12288992	153540528	-67515202851

$$j_0 = 1$$

$$j_1 = j - 744$$

$$j_2 = j^2 - 1488j + 159768$$

$$j_3 = j^3 - 2232j^2 + 1069956j - 36866976$$

Here the functions j_m, for $m \geq 0$ are of the form

$$j_m(\tau) = q^{-m} + \sum_{n \geq 1} c_m(n) q^n, \qquad (q = e(\tau) = e^{2\pi i \tau}) \tag{0.1}$$

They form a basis for the space $\mathbb{C}[j]$ of all weakly holomorphic modular forms of weight 0 and they have a generating function that goes back to Faber :

$$2\pi i \sum_{m \geq 0} j_m(z) q^m = \frac{j'(\tau)}{j(z) - j(\tau)}. \tag{0.2}$$

Note that this formal series converges when $\mathrm{Im}(\tau) > \mathrm{Im}(z)$.

m	$\mathrm{Tr}_{-3}(j_m)$	$\mathrm{Tr}_{-4}(j_m)$	$\mathrm{Tr}_{-7}(j_m)$
0	1/3	1/2	1
1	-248	492	-4119
2	53256	287244	16572393
3	-12288992	153540528	-67515202851

Let g_1 be the function given in terms of the usual modular forms E_4 and Δ and θ by

$$g_1(\tau) = \theta(\tau + \tfrac{1}{2}) \frac{E_4(4\tau)}{\Delta(4\tau)^{1/4}}.$$

Then

$$g_1(\tau) = q^{-1} - 2 + 248\, q^3 - 492\, q^4 + 4119\, q^7 + \cdots .$$

We have the following beautiful theorem of Zagier.

Theorem (D. Zagier)

Let

$$g(\tau) = q^{-1} + \sum_{0 \leq n \equiv 0,3(4)} \mathrm{Tr}_{-n}(j_1) q^n$$

Then $g \in M^!_{3/2}$.

What if $D > 0$?

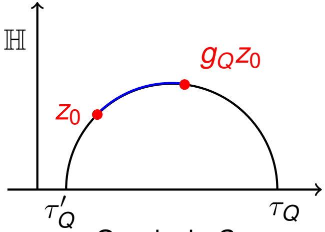
- The root τ_Q of $Q(\tau, 1) = 0$ for $Q \in \mathcal{Q}_D$ is a real quadratic irrationality.

What if $D > 0$?

- The root τ_Q of $Q(\tau, 1) = 0$ for $Q \in \mathcal{Q}_D$ is a real quadratic irrationality.
- Can we extend the definiton of "the value" of $j(\tau_Q)$, to these real quadratic irrationalities?

What if $D > 0$?

- The root τ_Q of $Q(\tau, 1) = 0$ for $Q \in \mathcal{Q}_D$ is a real quadratic irrationality.
- Can we extend the definiton of "the value" of $j(\tau_Q)$, to these real quadratic irrationalities?
- Recall for $D > 0$, non-square, to each $Q = [A, B, C] \in \mathcal{Q}_D$, we have associated a geodesic $S_Q = A|\tau|^2 + B\,\mathrm{Re}(\tau) + C \in \mathbb{H}$ and a closed geodesic $C_Q \in \Gamma\backslash\mathbb{H}$.



Geodesic S_Q

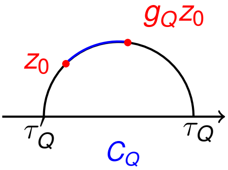
“Values” of modular functions at real quadratic irrationalities

- 2008, M. Kaneko and W. Duke, I., A. Toth independently defined real quadratic analogues of singular moduli through the cycle integrals of the j-function.

“Values” of modular functions at real quadratic irrationalities

- 2008, M. Kaneko and W. Duke, I., A. Toth independently defined real quadratic analogues of singular moduli through the cycle integrals of the j-function.
-

$$“j(\tau_Q)” := \int_{z_0}^{g_Q z_0} j_1(z) \frac{dz}{Az^2 + Bz + C} = \int_{C_Q} j_1(z) \frac{dz}{Q(z,1)}$$



$$Tr_D(j_1) := \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} j_1(\tau_Q)$$

Numerical approximations to the first few traces for $D > 0$.

m	$\mathrm{Tr}_5(j_m)$	$\mathrm{Tr}_8(j_m)$	$\mathrm{Tr}_{12}(j_m)$	$\mathrm{Tr}_{13}(j_m)$
0	.30634	.56109	.83840	.76060
1	-11.54175	-19.13749	-28.67973	-23.40950
2	-25.85662	-50.69769	-74.46436	-69.09778
3	-32.00316	-55.08485	-86.06819	-79.31283

Numerical approximations to the first few traces for $D > 0$.

m	$\mathrm{Tr}_5(j_m)$	$\mathrm{Tr}_8(j_m)$	$\mathrm{Tr}_{12}(j_m)$	$\mathrm{Tr}_{13}(j_m)$
0	.30634	.56109	.83840	.76060
1	-11.54175	-19.13749	-28.67973	-23.40950
2	-25.85662	-50.69769	-74.46436	-69.09778
3	-32.00316	-55.08485	-86.06819	-79.31283

m	$\mathrm{Tr}_{-3}(j_m)$	$\mathrm{Tr}_{-4}(j_m)$	$\mathrm{Tr}_{-7}(j_m)$
0	1/3	1/2	1
1	-248	492	-4119
2	53256	287244	16572393
3	-12288992	153540528	-67515202851

- The theory of these real quadratic analogues is still in its infancy.

- The theory of these real quadratic analogues is still in its infancy.
- Direct connections with arithmetic have not been found.

- The theory of these real quadratic analogues is still in its infancy.
- Direct connections with arithmetic have not been found.
- However, some connections with modular forms have.

Theorem (W. Duke, I., A. Toth and J. Bruinier, J. Funke, I.)

The function

$$h(\tau) = \sum_{D \geq 0} \mathrm{Tr}_D(j_1) q^D + g^*(\tau)$$

has weight 1/2 for $\Gamma_0(4)$, i.e. it transforms like $\theta(\tau)$.

The non holomorphic correction term $g^*(\tau)$ satisfy the differential equation

$$2iy^{1/2} \frac{d}{d\bar{\tau}} g^*(\tau) = \overline{g(\tau)}$$

where $g(\tau)$ is Zagier's function. Moreover h is harmonic.

$$\Delta_{1/2}(h) = 0$$

What can we say about the individual ”values” $j(\tau_Q)$ for $D > 0$?

Kaneko studied the numerical values of $j(\tau_Q)$ and made several remarkable observations.

What can we say about the individual ”values” $j(\tau_Q)$ for $D > 0$?

Kaneko studied the numerical values of $j(\tau_Q)$ and made several remarkable observations. For a general quadratic irrationality, among his many observations, we note the following boundedness conjecture.

What can we say about the individual "values" $j(\tau_Q)$ for $D > 0$?

Kaneko studied the numerical values of $j(\tau_Q)$ and made several remarkable observations. For a general quadratic irrationality, among his many observations, we note the following boundedness conjecture.

Conjecture (M. Kaneko)

Let $w \in \mathbb{Q}(\sqrt{D})$ be a real quadratic irrationality. Then

$$\mathrm{Re}(j^{nor}(w)) \in [706.324\ldots, 744] \quad \text{and} \quad \mathrm{Im}(j^{nor}(w)) \in (-1, 1)$$

What can we say about the individual "values" $j(\tau_Q)$ for $D > 0$?

Kaneko studied the numerical values of $j(\tau_Q)$ and made several remarkable observations. For a general quadratic irrationality, among his many observations, we note the following boundedness conjecture.

Conjecture (M. Kaneko)

Let $w \in \mathbb{Q}(\sqrt{D})$ be a real quadratic irrationality. Then

$$\mathrm{Re}(j^{nor}(w)) \in [706.324\ldots, 744] \quad \text{and} \quad \mathrm{Im}(j^{nor}(w)) \in (-1, 1)$$

where

$$706.324\ldots = j^{nor}\left(\frac{(1+\sqrt{5})}{2}\right), \text{ and } j^{nor}(w) := \frac{\sqrt{D}}{2\log\varepsilon} j(w).$$

Every $x \in \mathbb{R}$ has a unique '–' continued fraction expansion
$x = (b_0, b_1, b_2, \ldots)$ with $b_i \in \mathbb{Z}, \quad b_i \geq 2$ for $i \geq 1$

$$x = (b_0, b_1, b_2, \ldots) = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots}}}$$

and a unique '+' continued fraction expansion
$x = [a_0, a_1, a_2, \ldots]$ with $a_i \in \mathbb{Z}$ and $a_i \geq 1$ for $i \geq 1$.

$$x = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

w is a quadratic irrationality if and only if its '–' continued fraction expansion (or equivalently, its '+' continued fraction) is eventually periodic.
For $N \geq 2$, we have

$$w = \frac{N + \sqrt{N^2 - 4}}{2} = (N, N, N \ldots) = (\overline{N})$$

For the golden ratio we have

$$w = \frac{1 + \sqrt{5}}{2} = (2, 3, 3, 3, 3 \cdots) = (2, \bar{3})$$

$$w = \frac{1 + \sqrt{5}}{2} = [1, 1, 1, \ldots] = [\bar{1}] = [\bar{1}_2]$$

Theorem (Bengoechea, I.)

Let w be a real quadratic number with purely periodic negative continued fraction $w = (\overline{b_1, \ldots, b_n})$. Then we have

1. $\mathrm{Re}(j^{nor}(w)) \leq 744$.
2. *For any positive integer $N > 2$, the value $j^{nor}((\overline{N}))$ for the quadratic number $w = (\bar{N})$ is real and*

$$\lim_{N \to \infty} j^{nor}((\overline{N})) = 744.$$

3. *Let $(1+\sqrt{5})/2 = (2, \bar{3})$ be the golden ratio. If all the partial quotients b_r of w satisfy $b_r > 3M$ with $M = e^{55}$ then*

$$\mathrm{Re}(j^{nor}(w)) \geq j^{nor}((1+\sqrt{5})/2).$$

The generating function of $\mathrm{Tr}_d(j_m)$, $m = 0, 1, \ldots$

Let

$$F_d(\tau) := -\sum_{m=0}^{\infty} \mathrm{Tr}_d(j_m) q^m.$$

Then we have

Theorem (Duke-I-Toth)

For each $d > 0$ not a square the function F_d is a holomorphic modular integral of weight 2 with a rational period function:

$$F_d(\tau) - \tau^{-2} F_d(-1/\tau) = -\frac{\sqrt{d}}{\pi} \sum_{\substack{c<0<a \\ b^2-4ac=d}} (a\tau^2 + b\tau + c)^{-1}. \quad (0.3)$$

- Note that the period function has simple poles at certain real quadratic integers of discriminant d.
- This is the analog of a theorem of Borcherds who defined a function F_d which has singularities at CM points when $d < 0$.
- The appearance of a rational period function is perhaps not too surprising in view of the case $d = 0$.

$$\tau^{-2} E_2(-1/\tau) = E_2(\tau) + (2\pi i\,\tau)^{-1}. \tag{0.4}$$

- We can also define a function $F_Q(\tau)$ associated to one class in $[Q] \in \Gamma\backslash\mathcal{Q}_d$.

$$F_Q(\tau) := -\sum_{m=0}^{\infty} a_Q(m) q^m$$

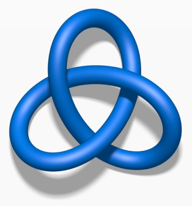
where

$$a_Q(m) := \sqrt{d} \int_{C_Q} j_m(z) \frac{dz}{Q(z,1)}.$$

- $F_Q(\tau)$ is again a modular integral with a rational period function.
- $F_Q(\tau)$ has an interesting application to topology.

An application to Topology

- Let $V = \{(z, w) \in \mathbb{C}^2 : z^3 - 27w^2 = 0\}$, and $T = V \cap S^3$, the trefoil knot.



An application to Topology

- Let $V = \{(z, w) \in \mathbb{C}^2 : z^3 - 27w^2 = 0\}$, and $T = V \cap S^3$, the trefoil knot.

- A remarkable fact: The homogeneous space

$$\mathrm{SL}(2, \mathbb{Z}) \backslash \mathrm{SL}(2, \mathbb{R})$$

is diffeomorphic to $S^3 - T$, the complement of the trefoil knot in the 3-sphere.

An application to Topology

$$E_2(\tau) = \frac{1}{2\pi i}\frac{\Delta'(\tau)}{\Delta(\tau)} = 1 - 24\sum \sigma(n)q^n$$

Dedekind studied the transformation law for the logarithm of

$$\Delta(\tau) = q\prod_{m\geq 1}(1-q^n)^{24}$$

An application to Topology

$$E_2(\tau) = \frac{1}{2\pi i}\frac{\Delta'(\tau)}{\Delta(\tau)} = 1 - 24\sum \sigma(n)q^n$$

Dedekind studied the transformation law for the logarithm of

$$\Delta(\tau) = q\prod_{m\geq 1}(1-q^n)^{24}$$

For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $c \neq 0$ we have

$$\log\Delta(\gamma\tau) - \log\Delta(\tau) = 6\log(-(c\tau+d)^2) + 2\pi i\Phi(\gamma).$$

We set $\Phi(\gamma) = 0$ if $c = 0$.

Here $\Phi(\gamma)$ is the Dedekind Symbol

$$\Phi(\gamma) = \frac{a+d}{c} - 12 sign(c) \cdot s(a,c),$$

$s(a,c)$ is the Dedekind sum defined for $\gcd(a,c) = 1$, $c \neq 0$ by

$$s(a,c) = \sum_{n=1}^{|c|-1} \left(\left(\frac{n}{c}\right)\right)\left(\left(\frac{na}{c}\right)\right).$$

$((x)) = 0$ if $x \in \mathbb{Z}$ and otherwise

$$((x)) = x - \lfloor x \rfloor - 1/2.$$

- The Rademacher Symbol $\Psi(\gamma)$ is

$$\Psi(\gamma) = \Phi(\gamma) - 3sign(c(a+d)).$$

- The Dedekind symbol $\Phi(\gamma)$ is not a conjugacy class invariant but the Rademacher symbol $\Psi(\gamma)$ is.

- The Rademacher Symbol $\Psi(\gamma)$ is

$$\Psi(\gamma) = \Phi(\gamma) - 3sign(c(a + d)).$$

- The Dedekind symbol $\Phi(\gamma)$ is not a conjugacy class invariant but the Rademacher symbol $\Psi(\gamma)$ is.
- For γ hyperbolic $\Psi(\gamma)$ is the homogenization of the Dedekind symbol $\Phi(\gamma)$,

$$\Psi(\gamma) = \lim_{n\to\infty} \frac{\Phi(\gamma^n)}{n}$$

- E. Ghys gave the beautiful result that the linking number of a “modular knot κ_γ” with the trefoil is given by the Rademacher symbol $\Psi(\gamma)$.

- Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ be a primitive hyperbolic element with an eigenvalue $\epsilon > 1$.

- Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ be a primitive hyperbolic element with an eigenvalue $\epsilon > 1$.
- Fix a $M \in G = \mathrm{SL}(2, \mathbb{R})$ so that

$$M^{-1}\gamma M = \begin{pmatrix} \epsilon & 0 \\ 0 & 1/\epsilon \end{pmatrix}.$$

- Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ be a primitive hyperbolic element with an eigenvalue $\epsilon > 1$.
- Fix a $M \in G = \mathrm{SL}(2, \mathbb{R})$ so that

$$M^{-1}\gamma M = \begin{pmatrix} \epsilon & 0 \\ 0 & 1/\epsilon \end{pmatrix}.$$

- Consider the 1-parameter subgroup

$$\varphi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

and the associated flow

$$\begin{aligned} G \times \mathbb{R} &\longmapsto G \\ (g, t) &\longmapsto g\varphi(t) \end{aligned}$$

- There is an induced flow ϕ^t on $\Gamma\backslash G$, called the geodesic flow.

- There is an induced flow ϕ^t on $\Gamma\backslash G$, called the geodesic flow.
- $$\phi^t : \Gamma g \longmapsto \Gamma g \varphi(t).$$

- There is an induced flow ϕ^t on $\Gamma\backslash G$, called the geodesic flow.
- $$\phi^t : \Gamma g \longmapsto \Gamma g \varphi(t).$$
- $\tilde{\gamma}(t) = M\varphi(t)$ is a periodic orbit in $\Gamma\backslash G$ with period $\log \epsilon$. i.e. $\tilde{\gamma}(t + \log \epsilon) = \tilde{\gamma}(t)$ in $\Gamma\backslash G$.

- There is an induced flow ϕ^t on $\Gamma\backslash G$, called the geodesic flow.
- $$\phi^t : \Gamma g \longmapsto \Gamma g\varphi(t).$$
- $\tilde{\gamma}(t) = M\varphi(t)$ is a periodic orbit in $\Gamma\backslash G$ with period $\log\epsilon$. i.e. $\tilde{\gamma}(t + \log\epsilon) = \tilde{\gamma}(t)$ in $\Gamma\backslash G$.
- This orbit depends only on the conjugacy class of γ.

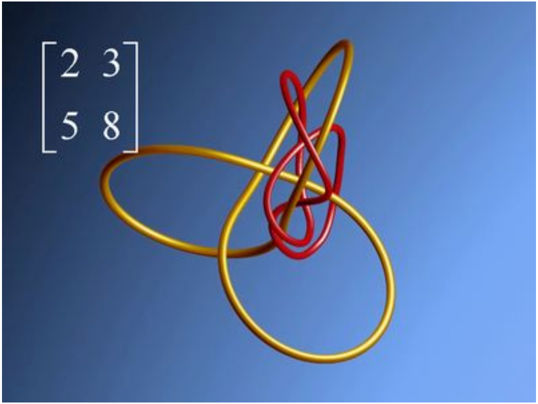
- There is an induced flow ϕ^t on $\Gamma\backslash G$, called the geodesic flow.
- $$\phi^t : \Gamma g \longmapsto \Gamma g \varphi(t).$$
- $\tilde{\gamma}(t) = M\varphi(t)$ is a periodic orbit in $\Gamma\backslash G$ with period $\log \epsilon$. i.e. $\tilde{\gamma}(t + \log \epsilon) = \tilde{\gamma}(t)$ in $\Gamma\backslash G$.
- This orbit depends only on the conjugacy class of γ.
- Periodic orbits of the geodesic flow correspond to conjugacy classes of hyperbolic elements.

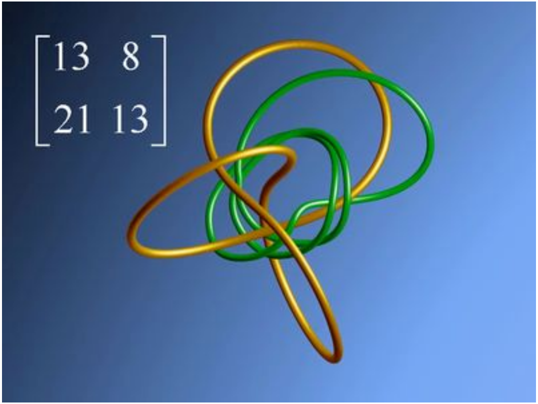
- There is an induced flow ϕ^t on $\Gamma\backslash G$, called the geodesic flow.
- $$\phi^t : \Gamma g \longmapsto \Gamma g\varphi(t).$$
- $\tilde{\gamma}(t) = M\varphi(t)$ is a periodic orbit in $\Gamma\backslash G$ with period $\log \epsilon$. i.e. $\tilde{\gamma}(t + \log \epsilon) = \tilde{\gamma}(t)$ in $\Gamma\backslash G$.
- This orbit depends only on the conjugacy class of γ.
- Periodic orbits of the geodesic flow correspond to conjugacy classes of hyperbolic elements.
- These orbits are also related to classes of indefinite binary quadratic forms

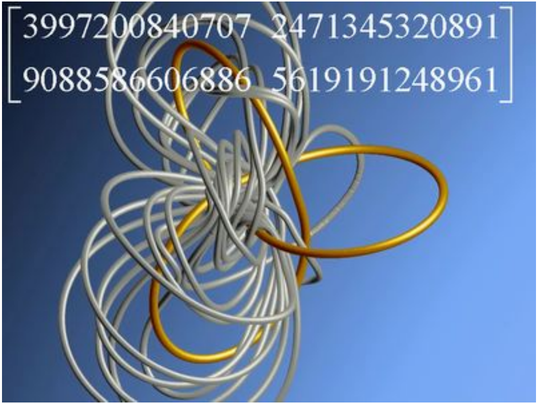
- Let $Q(X, Y) = AX^2 + BXY + CY^2$ be a quadratic forms of discriminant $D = B^2 - 4AC$,
- t, u are the smallest positive solutions of Pell's equation $t^2 - Du^2 = 4$, and
- $$\sigma_Q = \begin{pmatrix} \frac{t+Bu}{2} & Cu \\ -Au & \frac{t-Bu}{2} \end{pmatrix}.$$
- The association $Q \mapsto \sigma_Q$ sets up a bijection between elements of the class $\mathcal{C}$ of the quadratic form Q and the conjugacy class of σ_Q, which by abuse of notation will also be denoted by $\mathcal{C}$.

The image of the periodic orbit $\tilde{\gamma}(t)$ in $S^3 - T$ is a knot, κ_γ, called a modular knot.

The image of the periodic orbit $\tilde{\gamma}(t)$ in $S^3 - T$ is a knot, κ_γ, called a modular knot.



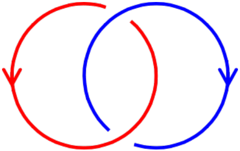

13 8
21 13



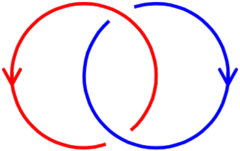
Pictures: ”*Mathematical Imagery*” by Jos Leys

Suppose that two oriented knots K_1 and K_2 do not intersect, and project them onto a plane in a generic way. These projections need not be disjoint.

Suppose that two oriented knots K_1 and K_2 do not intersect, and project them onto a plane in a generic way. These projections need not be disjoint.



Linking number 1
(*Pictures by Jim.belk*)



Linking number -1

Theorem (E. Ghys)

The linking number of the modular knot κ_γ and the trefoil is given by the Rademacher symbol.

$$\mathrm{Lk}(\kappa_\gamma, T) = \Psi(\gamma)$$

The Dedekind Symbol has other representations.

- For $c \neq 0$, let

$$R(\gamma, z) = 6\log(-(cz+d)^2) + 2\pi i \Phi(\gamma) = \log\Delta(\gamma\tau) - \log\Delta(\tau)$$

  Then

$$\Phi(\gamma) = \frac{1}{2\pi} \lim_{y\to\infty} \operatorname{Im} R(\gamma, iy).$$

- It also has a representation in terms of the special value of an $L-$ function associated to $E_2(z)$.

$$L(s, a/c) = \sum_{n\geq 1} \sigma(n) e(\tfrac{a}{c} n) n^{-s}$$

- At the end of his paper, *Knots and Dynamics*, E. Ghys mentions the problem of interpreting the linking number between two modular knots.

- At the end of his paper, *Knots and Dynamics*, E. Ghys mentions the problem of interpreting the linking number between two modular knots.
- We will look at this problem by giving an appropriate generalization of the Dedekind symbol.

- For an hyperbolic element $\sigma \in \mathcal{C}$ let

$$a_{\mathcal{C}}(m) = \sqrt{D'} \int_{z_0}^{\sigma z_0} j_m(z) \tfrac{dz}{Q_\sigma(z)}.$$

- Define for $\alpha \in \mathbb{Q}$ the Dirichlet series

$$L_{\mathcal{C}}(s, \alpha) = \sum_{n \geq 1} a_{\mathcal{C}}(n) e(n\alpha) n^{-s}, \qquad (0.5)$$

- Then $L_{\mathcal{C}}(s, a/c)$ converges for $\mathrm{Re}(s) > 9/4$, has a meromorphic continuation to $s > 0$ and is holomorphic at $s = 1$.

- We define a new Dedekind symbol,
  Dedekind symbol associate to the conjugacy class $\mathcal{C}$,

$$\Phi_{\mathcal{C}}(\gamma) := -\frac{1}{\pi^2} \operatorname{Re} L_{\mathcal{C}}(1, \tfrac{a}{c}).$$

Then we have

- Theorem [DIT] $\Phi_{\mathcal{C}}(\gamma)$ is an integer.

- For simplicity we assume $\sigma \in \mathcal{C}_\sigma$ is conjugate to its inverse, ie. $\sigma \sim \sigma^{-1}$. Then the modular knot κ_σ is null-homologous in $S^3 - T$. For two such knots κ_σ and κ_γ we have

Theorem (DIT)

Let

$$\Psi_\sigma(\gamma) = \lim_{n\to\infty} \frac{\Phi_{\mathcal{C}_\sigma}(\gamma^n)}{n}.$$

Then the limit exists and

- For simplicity we assume $\sigma \in \mathcal{C}_\sigma$ is conjugate to its inverse, ie. $\sigma \sim \sigma^{-1}$. Then the modular knot κ_σ is null-homologous in $S^3 - T$. For two such knots κ_σ and κ_γ we have

Theorem (DIT)

Let

$$\Psi_\sigma(\gamma) = \lim_{n\to\infty} \frac{\Phi_{\mathcal{C}_\sigma}(\gamma^n)}{n}.$$

Then the limit exists and

$$\Psi_\sigma(\gamma) = Lk(\kappa_\gamma, \kappa_\sigma).$$

THANK YOU!